SCIENTIFIC OASIS

Decision Making Advances

Journal homepage: www.dma-journal.org
ISSN: 2956-2384

A Distance Measure Between Fuzzy Implications

Anestis G. Hatzimichailidis[1*]

1 Department of Agricultural Biotechnology and Oenology, Democritus University of Thrace, Drama, Greece

ARTICLE INFO

Article history:
Received 20 September 2023
Received in revised form 19 April 2024
Accepted 8 June 2024
Available online 20 June 2024

Keywords:
Distance Measure; Similarity Measure; Fuzzy Implication.

ABSTRACT

In this paper, we study a distance measure between fuzzy implications. The proposed distance measure is normalized and, therefore, gives rise to the corresponding similarity measure. The existence of a distance measure of fuzzy implication and the quantification of the similarity of these is very important in applications.

1. Introduction

Various distance and similarity measures for fuzzy sets, as well as their applications, have been proposed in the literature [1-14]. A theoretical usefulness of similarity measures is that it enables us to classify fuzzy implications into equivalence classes, based on their degree of similarity [15]. In addition, to a specific problem, similarity classifications are a useful tool because help us to calculate the degree of similarity of the behavior of these.

Balopoulos et al. [1], proposed new families of normalized distance measures and its dual families of similarity measures between binary fuzzy operators. The family of distance measures has emerged from the transformation of a metric distance between matrices by using fuzzy implications. These definitions, they on the availability of two particular finite fuzzy sets, which, in the case of fuzzy implications, may be viewed as a set of predicates and a set of antecedents.

The present study uses a metric distance to measure the "distance" between two fuzzy implications, which is not subject to the above restrictions. The corresponding similarity measure of this metric distance, which we use in this paper, can used to determine the degree of similarity and/or the degree of inclusion between two fuzzy implications.

The layout of this paper is as follows. In Section 2, we recall some basic concepts that are more or less known in the literature. Section 3 studies a metric measure for fuzzy implications and its dual

* *Corresponding author.*
E-mail address: hatzane@emt.ihu.gr

https://doi.org/10.31181/dma21202431

similarity measure. Moreover, Section 4 gives a numerical example. Section 5 summarizes the contribution of this work.

2. Mathematical Background – Preliminaries

This section recalls some basic concepts associated with metric distances and fuzzy implications. All basic concepts, that we are going to use in this paper, can be found in [6, 16-20].

2.1 Metric Distances - Basic Notations

We recall that a metric distance in a set X is a real function $d: X \times X \to \mathbb{R}$, which satisfies, $\forall a, b, c \in X$ the following three conditions:

i. *Reflexivity* – $d(a, b) = 0 \Leftrightarrow a = b$.
ii. *Symmetry* – $d(a, b) = d(b, a)$.
iii. *Triangle inequality* – $d(a, c) + d(c, b) \geq d(a, b)$.

Note that from the above arises $d(a, b) \geq 0$. Various metric distances, involving fuzzy sets, have been proposed in the literature. Let two fuzzy subsets F_1 and F_2, of a universe set X, with membership functions $\mu_{F1}(.)$ and $\mu_{F2}(.)$, respectively. We have:

$$\boldsymbol{d_1(F_1, F_2) = \int_X |\mu_{F1}(x) - \mu_{F2}(x)| dx,} \tag{1}$$

$$\boldsymbol{d_\infty(F_1, F_2) = \vee_{x \in X} |\mu_{F1}(x) - \mu_{F2}(x)|,} \tag{2}$$

$$\boldsymbol{d_p(F_1, F_2) = (\int_X |\mu_{F1}(x) - \mu_{F2}(x)|^p dx)^{1/p}, p \geq 1,} \tag{3}$$

$$\boldsymbol{d_{ess.sup}(F_1, F_2) = inf\{k \in \mathbb{R}: \lambda(\{x \in X: |\mu_{F1}(x) - \mu_{F2}(x)| > k\}) = 0}. \tag{4}$$

The family of distances from Eqs. (1)–(3) are called the Minkowski distances. In Eq. (3), p is a parameter satisfying $1 \leq p \leq +\infty$. When $p = 1$, d_1 is called the Hamming distance of fuzzy sets. When $p = 2$, d_2 is called the Euclidean distance. When $p = \infty$, d_∞ is called the Chebyshev distance of fuzzy sets. The distance in Eq. (4) is called the essential supremum. It is known that $\lim_{p \to \infty} d_p(F_1, F_2) = d_{ess.sup}(F_1, F_2)$ and $\lim_{p \to \infty} d_p(F_1, F_2) \leq d_\infty(F_1, F_2)$ both provided that $\lambda(X) < \infty$. In that case, it is $d_{ess.sup}(F_1, F_2) \leq d_\infty(F_1, F_2)$.

2.2 Fuzzy Implications - Basic Notations

A fuzzy implication is a function $\sigma: [0,1] \times [0,1] \to [0,1]$, which for any truth values $x, y \in [0,1]$ of propositions *P* and *Q*, respectively, gives the truth value $\sigma(x, y)$ of the conditional proposition: "if *P* then *Q* ".

One way of defining an implication operator $\to$ in classical logic is by using:

$$\boldsymbol{x \to y = \neg x \vee y,\ x, y \in \{0, 1\}}. \tag{5}$$

This formula can also be rewritten based on the law of absorption of negation in classical logic, as either (for all $x, y \in \{0,1\}$):

$$\boldsymbol{x \to y = max\{x \in \{0,1\}: a \wedge x \leq b\}}, \tag{6}$$

or

$$\boldsymbol{x \to y = \neg x \vee (x \wedge y)}, \tag{7}$$

or

$$\boldsymbol{x \to y = (\neg x \wedge \neg y) \vee y}. \tag{8}$$

Fuzzy logic extensions of the previous four formulas $\forall a, b \in [0,1]$ are:

$$\boldsymbol{\sigma_S(x,y) = S(n(x),y)}, \tag{9}$$

$$\boldsymbol{\sigma_R(x,y) = sup\{a \in [0,1]: T(x,a) \leq y\}}, \tag{10}$$

$$\boldsymbol{\sigma_{QL}(x,y) = S(n(x),T(x,y))}, \tag{11}$$

$$\boldsymbol{\sigma_D(x,y) = S\big(T\big(n(x),n(y)\big),y\big)}. \tag{12}$$

where *S*, *T*, and *n* denote a t-conorm, a t-norm, and a fuzzy negation on [0,1], respectively, and the triple <*S*, *T*, *n*> is required to satisfy the De Morgan laws $n(T(x,y)) = S\big(n(x),n(y)\big)$ and $n(S(x,y)) = T\big(n(x),n(y)\big)$, $\forall x,y \in [0,1]$. Note that fuzzy implications obtained from Eq. (12) are usually referred to as D-implications [5, 19].

3. Measuring the Distance between Fuzzy Implications

3.1. A Distance Measure between Fuzzy Implications

Let *FS*(*X*) denote the set of fuzzy sets in *X* and let $\mathfrak{I}$ denote the set of fuzzy implications. In the following, for short, we denote $\sigma(a,\beta)$, where $a,\beta \in [0,1]$, instead of $\sigma(A(x),B(y))$, where $A,B \in FS(X)$. Let $\sigma_1,\sigma_2 \in \mathfrak{I}$. We calculate the space, which is located between the graphic representations, of these two fuzzy implications, using the function $d_h(\sigma_1,\sigma_2) = \iint_R |\sigma_1(a,\beta) - \sigma_2(a,\beta)| da d\beta$, where $R = [0,1] \times [0,1] = \{(a,\beta) | a,\beta \in [0,1]\}$.

The following proposition introduces a distance measure between fuzzy implications.

Proposition 1: *Let* $\sigma_1,\sigma_2 \in \mathfrak{I}$. *A distance measure function* $d_h: \mathfrak{I} \times \mathfrak{I} \to [0,+\infty)$ *is given by* $d_h(\sigma_1,\sigma_2) = \iint_R |\sigma_1(a,\beta) - \sigma_2(a,\beta)| da d\beta$, *where* $R: 0 \leq a \leq 1, 0 \leq \beta \leq 1$.

Proof of Proposition 1: The function d_h must be satisfies the three conditions of metric distance:

i. *Reflexivity property* – $d_h(\sigma_1,\sigma_2) = 0 \Leftrightarrow \sigma_1(a,\beta) = \sigma_2(a,\beta), \forall a,\beta \in [0,1]$.
ii. *Symmetric property* – $d_h(\sigma_2,\sigma_1) \Leftrightarrow \iint_R |\sigma_1(a,\beta) - \sigma_2(a,\beta)| da d\beta = \iint_R |\sigma_2(a,\beta) - \sigma_1(a,\beta)| da d\beta \Leftrightarrow |\sigma_1(a,\beta) - \sigma_2(a,\beta)| = |\sigma_2(a,\beta) - \sigma_1(a,\beta)|$.
iii. *Triangle inequality*:

$$\boldsymbol{d_h(\sigma_1,\sigma_2) + d_h(\sigma_2,\sigma_3) = \iint_R |\sigma_1(a,\beta) - \sigma_2(a,\beta)| da d\beta + \iint_R |\sigma_2(a,\beta) - \sigma_3(a,\beta)| da d\beta}, \tag{13}$$

and

$$\boldsymbol{d_h(\sigma_1,\sigma_3)=\iint_R|\sigma_1(a,\beta)-\sigma_3(a,\beta)|dad\beta}, \tag{14}$$

with:

$$\boldsymbol{|\sigma_1(a,\beta)-\sigma_2(a,\beta)|+|\sigma_2(a,\beta)-\sigma_3(a,\beta)|\geq|\sigma_1(a,\beta)-\sigma_3(a,\beta)|}. \tag{15}$$

Comparing Eq. (13), Eq. (14), and Eq. (15), we conclude that $d_h(\sigma_1,\sigma_2)+d_h(\sigma_2,\sigma_3)\geq d_h(\sigma_1,\sigma_3)$. Thus, Proposition 1 is proven.

Proposition 2: *Let $d_h:\mathfrak{I}\times\mathfrak{I}\to[0,+\infty)$ be the above distance measure between fuzzy implications. Then the following statements are true:*

- *i.* $d_h(\sigma_1,\sigma_2)\geq 0$ *for every* $\sigma_1,\sigma_2\in\mathfrak{I}$.
- *ii.* *Generalized triangle inequality* – $d_h(\sigma_1,\sigma_n)\leq d_h(\sigma_1,\sigma_2)+d_h(\sigma_2,\sigma_3)+\cdots+d_h(\sigma_{n-1},\sigma_n)$ *for every* $\sigma_1,\sigma_2,\dots,\sigma_n\in\mathfrak{I}$.
- *iii.* $|d_h(\sigma_1,\sigma_3)-d_h(\sigma_2,\sigma_3)|\leq d_h(\sigma_1,\sigma_2)$ *for every* $\sigma_1,\sigma_2,\sigma_3\in\mathfrak{I}$.
- *iv.* $|d_h(\sigma_1,\sigma_2)-d_h(\sigma_3,\sigma_4)|\leq d_h(\sigma_1,\sigma_3)+d_h(\sigma_2,\sigma_4)$ *for every* $\sigma_1,\sigma_2,\sigma_3,\sigma_4\in\mathfrak{I}$.

Proof of Proposition 2: The function d_h must be satisfies the three conditions of metric distance:

- i. In the triangle inequality $d_h(\sigma_1,\sigma_2)+d_h(\sigma_2,\sigma_3)\geq d_h(\sigma_1,\sigma_3)$ we place σ_1 where σ_3 and using the symmetric property we take that $d_h(\sigma_1,\sigma_2)\geq 0$
- ii. Proof of this statement immediately results from the triangle inequality, using the inductive method.
- iii. From $d_h(\sigma_1,\sigma_2)+d_h(\sigma_2,\sigma_3)\geq d_h(\sigma_1,\sigma_3)$ for every $\sigma_1,\sigma_2,\sigma_3\in\mathfrak{I}$, we have:

$$\boldsymbol{d_h(\sigma_1,\sigma_3)-d_h(\sigma_2,\sigma_3)\leq d_h(\sigma_1,\sigma_2)}. \tag{16}$$

In Eq. (12), we place σ_1 where σ_2 and σ_2 where σ_1:

$$\boldsymbol{d_h(\sigma_2,\sigma_3)-d_h(\sigma_1,\sigma_3)\leq d_h(\sigma_2,\sigma_1)}. \tag{17}$$

Comparing Eq. (16) and Eq. (17) we conclude that $|d_h(\sigma_1,\sigma_3)-d_h(\sigma_2,\sigma_3)|\leq d_h(\sigma_1,\sigma_2)$.

- iv. This statement is proven in a similar manner, using the generalized triangular inequality. Thus, Proposition 2 is proven.

3.2. A Dual Similarity Measure for Fuzzy Implications

We introduce a similarity measure for binary fuzzy operators. The distance measure for binary fuzzy operators is normalized and therefore gives rise to the corresponding similarity measure.

Proposition 3: *The degree of similarity between two fuzzy implications $\sigma_1,\sigma_2\in\mathfrak{I}$ is defined as $S_h(\sigma_1,\sigma_2)=1-d_h(\sigma_1,\sigma_2)=1-\iint_R|\sigma_1(a,\beta)-\sigma_2(a,\beta)|dad\beta$, where $R:0\leq a,\beta\leq 1$.*

The similarity measures are continuous, reflexive, and symmetric in their arguments, take values in $[0,1]$, and are exactly 1 for $\sigma_1=\sigma_2$.

4. Numerical Example

In fuzzy systems applications, the predominant practice, known as the Mamdani method, is to employ fuzzy products instead of implications. The fuzzy products most commonly used in applications are ($\forall a, \beta \in [0,1]$)

i. *Mamdani rule* – $\sigma_M(a, \beta) = \min(a, \beta)$.
ii. *Larsen rule* – $\sigma_{La}(a, \beta) = a\beta$.

Product-based distance measures for fuzzy sets are similar to traditional measures in that they emphasize plausibility values. They simply increase the available variety. We refer to $\sigma_{\mathrm{M}}, \sigma_{\mathrm{La}}$, as the "engineering implications", contrary to the fuzzy implications [1].

In this section, we refer to and employ four fuzzy implications and two engineering implications. All these are listed here to avoid repetition ($\forall a, \beta \in [0,1]$):

i. *Reichenbach* – $\sigma_{Re}(a, \beta) = 1 - a + a\beta$.
ii. *Lukasiewicz* – $\sigma_L(a, \beta) = \min(1, 1 - a + \beta)$.
iii. *Kleene-Dienes* – $\sigma_{KD}(a, \beta) = \max(1 - a, \beta)$.
iv. *Gödel* – $\sigma_G(a, \beta) = \begin{cases} 1, for\ a \le \beta \\ \beta, for\ a > \beta \end{cases}$.
v. *Klir-Yuan* – $\sigma_{KY}(a, \beta) = 1 - a + a^2\beta$.
vi. *Mamdani* – $\sigma_M(a, \beta) = min(a, \beta)$.
vii. *Larsen* – $\sigma_{La}(a, \beta) = a\beta$.

Let the fuzzy implications $\sigma_{Re}(a, \beta), \sigma_{KY}(a, \beta)$, where $a, \beta \in [0,1]$. We calculated the distance $d_h(\sigma_{Re}, \sigma_{KY})$, as $d_h(\sigma_{Re}, \sigma_{KY}) = \iint_R |\sigma_{Re}(a, \beta) - \sigma_{KY}(a, \beta)| da d\beta$, with $R: 0 \le a, \beta \le 1$, and $d_h(\sigma_{Re}, \sigma_{KY}) = \iint_R |(1 - a + a\beta) - (1 - a + a^2\beta)| da d\beta = 0.0833$.

The degree of similarity between these two fuzzy implications is $S_h(\sigma_{Re}, \sigma_{KY}) = 1 - d_h(\sigma_{Re}, \sigma_{KY}) = 0.9167$. In a similar manner, for all pairs of operators from $\sigma_{Re}, \sigma_L, \sigma_{KD}, \sigma_G, \sigma_{KY}, \sigma_M, \sigma_{La}$, we measured distance and similarity, where $0 \le a, \beta \le 1$. We reported all pairwise similarity values in Table 1. We do not tabulate the corresponding distance values, as they are simply complements with respect to 1.

Table 1
Fuzzy implications and their similarity values

$S_h(.,.)$	σ_{Re}	σ_L	σ_{KD}	σ_G	σ_{KY}	σ_M	σ_{La}
σ_{Re}	1	0.9167	0.9167	0.8337	0.9167	0.5833	0.5000
σ_L	0.9167	1	0.8333	0.8334	0.8333	0.5000	0.4167
σ_{KD}	0.9167	0.8333	1	0.8320	0.947	0.6667	0.5833
σ_G	0.8337	0.8334	0.8320	1	0.7817	0.6666	0.5330
σ_{KY}	0.9167	0.8333	0.947	0.7817	1	0.6583	0.5833
σ_M	0.5833	0.5000	0.6667	0.6666	0.6583	1	0.9167
σ_{La}	0.5000	0.4167	0.5833	0.5330	0.5833	0.9167	1

From the limited numerical evidence presented here, a first comment, all $S_h(.,.)$ for all pairs of operators from $\sigma_{Re}, \sigma_L, \sigma_{KD}, \sigma_G, \sigma_{KY}$ differ little from each other and a lot from σ_M, σ_{La}. The most important distinguishing factor among the operators considered is whether they are fuzzy implications or fuzzy products. Therefore, if the proposed similarity measure is to be meaningful and useful, it is important to be able to distinguish fuzzy implications from fuzzy products. These numerical results show that the proposed similarity measure agrees with intuition, as is required from any fuzzy tool.

5. Conclusions

Generally is accepted that fuzzy sets are much more efficient at representing uncertainty in real-life problems. In this framework, we have proposed a distance measure between fuzzy implications and its dual similarity measure. The proposed distance measure and its dual similarity measure refer to fuzzy implications $\sigma(A(x), B(y))$, where A and B are infinite sets. The proposed measures between fuzzy implications, give satisfactory results, compared with the most commonly used metrics, as shown by the given numerical example. The distance measures are application-oriented. Therefore, the existence of distance measures and similarity measures between fuzzy implications helps us to classify them. Finally, we believe that the similarity measure for fuzzy implications provides a convenient means for their classification in the context of any given problem.

Conflicts of Interest
The author declares no conflicts of interest.

References

[1] Balopoulos, V., Hatzimichailidis, A. G., & Papadopoulos, B. K. (2007). Distance and similarity measures for fuzzy operators. *Information Sciences, 177*(11), 2336-2348. https://doi.org/10.1016/j.ins.2007.01.005.

[2] Ejegwa, P. A. (2020). Distance and similarity measures for Pythagorean fuzzy sets. *Granular Computing, 5*, 225–238. https://doi.org/10.1007/s41066-018-00149-z.

[3] Hatzimichailidis, A. G., & Papadopoulos, B. K. (2008). Similarity Classes on Fuzzy Implications. *Journal of Multiple-Valued Logic and Soft Computing, 14*(1-2), 105-117.

[4] Hatzimichailidis, A. G., & Kaburlasos, V. G. (2008). A novel fuzzy implication stemming from a fuzzy lattice inclusion measure. In: *Proceedings of the Lattice-Based Modeling Workshop, in conjunction with The Sixth International Conference on Concept Lattices and their Applications*, Olomouc, The Czech Republic, October 21-23, pp. 59-66.

[5] Hatzimichailidis, A. G, Papakostas, G. A., & Kaburlasos, V. G. (2016). A Distance Measure based on Fuzzy D-implications: Application in Pattern Recognition. *British Journal of Mathematics & Computer Science, 14*(3), 1-14. https://doi.org/10.9734/BJMCS/2016/23993.

[6] Hatzimichailidis, A. G., Papakostas, G. A., & Kaburlasos, V. G. (2016). On Constructing Distance and Similarity Measures based on Fuzzy Implications. In: *Handbook of Fuzzy Sets Comparison - Theory, Algorithms and Applications* (pp. 1-21), Book Series: Gate to Computer Science and Research, 6.

[7] Paik, B., & Mondal, S.K. (2020). A distance-similarity method to solve fuzzy sets and fuzzy soft sets based decision-making problems. *Soft Computing, 24*, 5217–5229. https://doi.org/10.1007/s00500-019-04273-z.

[8] Rani, V., & Kumar, S. (2023). MCDM Method for Evaluating and Ranking the Online Shopping Websites Based on a Novel Distance Measure under Intuitionistic Fuzzy Environment. *Operations Research Forum, 4*, 78. https://doi.org/10.1007/s43069-023-00258-9.

[9] Singh, S., & Singh, K. (2023). Novel fuzzy similarity measures and their applications in pattern recognition and clustering analysis. *Granular Computing, 8*, 1715-1737. https://doi.org/10.1007/s41066-023-00393-y.

[10] Sivadas, A., & John, S.J. (2023). Distance and similarity measures for (*p*, *q*)-fuzzy sets and their application in assessing common lung diseases. *SN Applied Science, 5*, 372. https://doi.org/10.1007/s42452-023-05580-9.

[11] Zeng, W., Ma, R., Li, D., Yin, Q., & Xu, Z. (2022). Distance Measure of Hesitant Fuzzy Sets and its Application in Image Segmentation. *International Journal of Fuzzy Systems, 24*, 3134–3143. https://doi.org/10.1007/s40815-022-01328-6.

[12] Zhang, C., & Fu, H. (2006). Similarity measures on three kinds of fuzzy sets. *Pattern Recognition Letters, 27*(12), 1307-1317. https://doi.org/10.1016/j.patrec.2005.11.020.

[13] Zhu, S., & Liu, Z. (2023). Distance measures of picture fuzzy sets and interval-valued picture fuzzy sets with their applications. *AIMS Mathematics, 8*(12), 29817–29848. https://doi.org/10.3934/math.20231525.

[14] Zwick R., Carlstein E., & Budescu D. (1987). Measures of similarity among fuzzy sets: A comparative analysis. I*nternational Journal of Approximate* Reasoning, *1*(2), 221–242. https://doi.org/10.1016/0888-613X(87)90015-6.

[15] Hatzimichailidis, A. G., & Papadopoulos, B. K. (2008). Ordering relation of fuzzy implications. *Journal of Intelligent & Fuzzy Systems, 19*(3), 189–195.

[16] Bustince, H., Burillo, P., & Soria, F. (2003). Automorphisms, negations and implication operators. *Fuzzy Sets Systems*, *134*(2), 209-229. https://doi.org/10.1016/S0165-0114(02)00214-2.

[17] Hatzimichailidis, A. G., & Papadopoulos, B. K. (2007). L-fuzzy Sets and Intuitionistic Fuzzy Sets. In: *Computational Intelligence Based on Lattice Theory* (pp. 325-339), Studies in Computational Intelligence, Vol. 67, Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-540-72687-6_16.

[18] Klir, G., & Yuan, B. (1995). *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, 1st Ed. Prentice Hall Press: Upper Saddle River, NJ, USA.

[19] Mas, M., Monserrat, M., Torrens, J., & Trillas, E. (2007). A survey on fuzzy implication functions. *IEEE Transactions on Fuzzy Systems, 15*(6), 1107-1121. https://doi.org/10.1109/TFUZZ.2007.896304.

[20] Nguyen, H. T., Walker, C., & Walker, E. A. (2018). *A First Course in Fuzzy Logic*, 4th ed. Chapman and Hall/CRC, NY, USA. https://doi.org/10.1201/9780429505546.